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Adjunctions and limits for double and multiple categories

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Genova

Based on a series of joint papers with R. Paré

1. Introduction

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- - in higher (also infinite) dimension, we have:

  – weak multiple categories of cubical spans or cospans,

  – chiral multiple categories of spans and cospans, etc.

  with all multiple limits and colimits

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F\colon \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \qquad F(X) = X \otimes A, \quad G(Y) = \mathrm{Hom}(A, Y)$$

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Extending F, G to $\mathrm{Rel}\mathbf{Ab}$ (locally ordered 2-category) we get:

$$F' = \mathrm{Rel}(F)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad F'(vu) \leqslant F'(v).F'(u) \quad (colax)$$

$$G' = \mathrm{Rel}(G)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad G'(v).G'(u) \leqslant G'(vu) \quad (lax)$$

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Extending the adjunction makes problems

- We cannot compose F' and G'
- What do we make of unit and counit?

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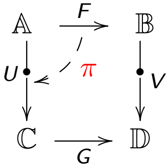
F'' and G'' are *orthogonal adjoints* in $\mathbb{D}\mathrm{bl}$ ($\mathbb{X} = \mathbb{A} = \mathbb{R}\mathrm{el}\mathbf{Ab}$)

$$\begin{array}{ccc} \mathbb{X} & = & \mathbb{X} \\ F''\,\big\downarrow^{\bullet} & \eta & \big\| \\ \mathbb{A} & \xrightarrow[G'']{} & \mathbb{X} \end{array} \qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{G''} & \mathbb{X} \\ \big\| & \varepsilon & {}^{\bullet}\big\downarrow\, F'' \\ \mathbb{A} & = & \mathbb{A} \end{array}$$

$$\frac{\eta}{\varepsilon} = 1_{F''}$$

$$\varepsilon \,|\, \eta = e_{G''}$$

4. The double category $\mathbb{D}\mathrm{bl}$ of weak double categories



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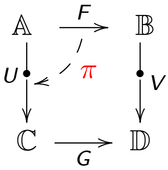

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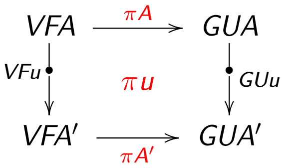

(for $u\colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{A}$)

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$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{B} \\ {\scriptstyle U}\,\big\downarrow\!\bullet & \swarrow\!\!\!{\scriptstyle \pi} & \bullet\!\big\downarrow\,{\scriptstyle V} \\ \mathbb{C} & \xrightarrow[G]{} & \mathbb{D} \end{array}$$

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$$\begin{array}{ccc} VFA & \xrightarrow{\pi A} & GUA \\ {\scriptstyle VFu}\,\big\downarrow\!\bullet & \pi u & \bullet\!\big\downarrow\,{\scriptstyle GUu} \\ VFA' & \xrightarrow[\pi A']{} & GUA' \end{array} \qquad (\text{for } u\colon A \,{\to\!\!\!\!\bullet}\, A' \text{ in } \mathbb{A})$$

Coherence conditions (*besides naturality*) for A and $w = u \otimes v$ in $\mathbb{A}$:

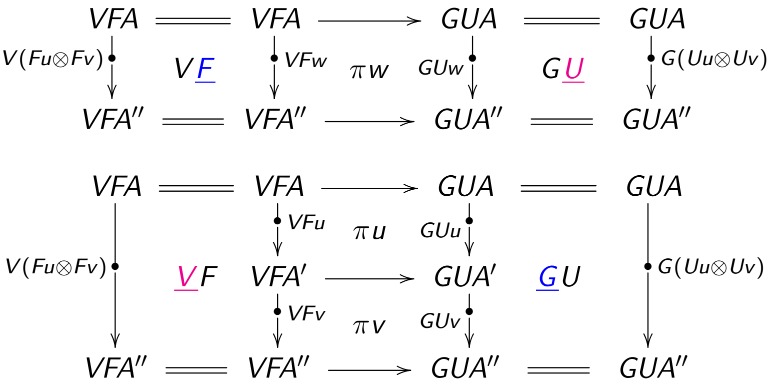
$$(V\underline{F}A \mid \pi e_A \mid G\underline{U}A) = (\underline{V}FA \mid e_{\pi A} \mid \underline{G}UA)$$

$$(V\underline{F}(u, v) \mid \pi w \mid G\underline{U}(u, v)) = (\underline{V}(Fu, Fv) \mid (\pi u \otimes \pi v) \mid \underline{G}(Uu, Uv))$$

5. The second coherence condition (for vertical composition)

- based on the laxity comparisons $\underline{F}$, $\underline{G}$ (of F, G) and the colaxity comparisons $\underline{U}$, $\underline{V}$ (of U, V), for $w = u \otimes v$ in $\mathbb{A}$

$$(V\underline{F}(u,v) \,|\, \pi w \,|\, G\underline{U}(u,v)) = (\underline{V}(Fu, Fv) \,|\, (\pi u \otimes \pi v) \,|\, \underline{G}(Uu, Uv))$$



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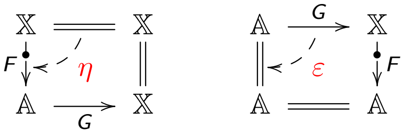

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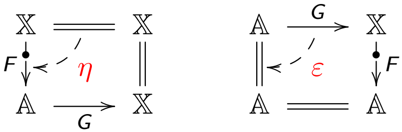

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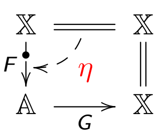

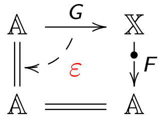

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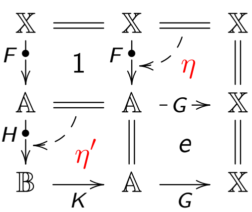

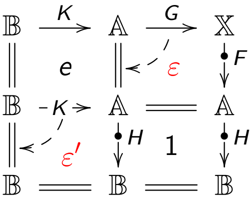

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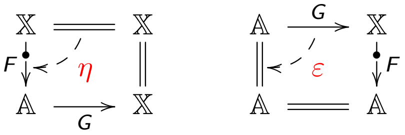

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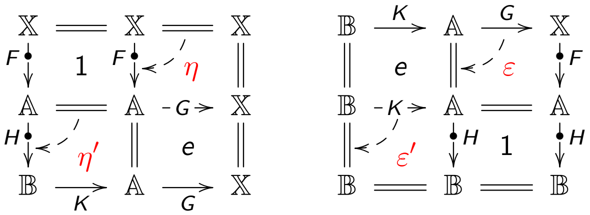

(Pseudo-lax adjunction: in the 2-category $\mathrm{Lx}\mathbf{Dbl} = \mathbf{Hor}\mathbb{D}\mathrm{bl}$)

(Colax-pseudo adjunction: in the 2-category $\mathrm{Cx}\mathbf{Dbl} = \mathbf{Ver}^{*}\mathbb{D}\mathrm{bl}$)

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A family of relations $u_i \colon A_i \nrightarrow B_i$ $(i \in I)$ has an obvious product

$$u \colon A \nrightarrow B, \qquad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

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The existence of functorial double products in $\mathbb{R}\mathrm{el}\mathbf{Ab}$ means that:
- the category of horizontal arrows (i.e. **Ab**) has products,
- the category of vertical arrows and double cells has products,
- these solutions agree w.r.t. vertical (co)domain and identities.

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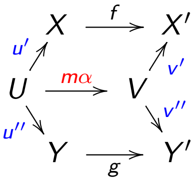
on a category **C** with pullbacks and pushouts



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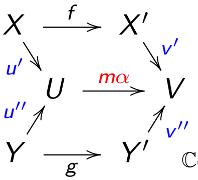

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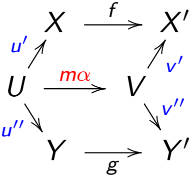

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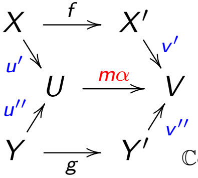

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The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

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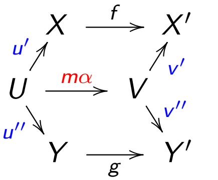

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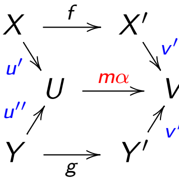

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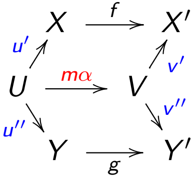

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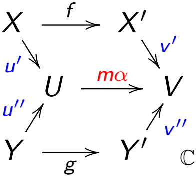

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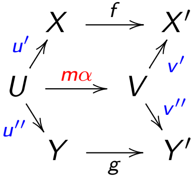

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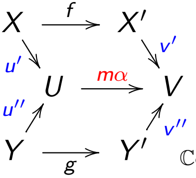

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($\mathbb{S}\mathrm{pan}\mathbf{C}$ has all double (co)limits, if $\mathbf{C}$ has (co)limits)

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- if **C** is abelian, the po/pb adjunction is strong (left and right)

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(with the composed comparisons)

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- if **C** is abelian, the po/pb adjunction is strong (left and right)
$\to$ idempotent lax monad T on $\mathbb{S}\mathrm{pan}\mathbf{C}$: $\mathbb{A}\mathrm{lg}(T) = \mathbb{R}\mathrm{el}\mathbf{C}$
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($\mathbb{R}\mathrm{el}\mathbf{Set} = \mathbb{A}\mathrm{lg}(T')$ for the jointly-monic monad T' on $\mathbb{S}\mathrm{pan}\mathbf{Set}$)

10. Weak multiple categories

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- an n-dimensional cube is a functor $x\colon \vee^n \to \mathbf{Set}$

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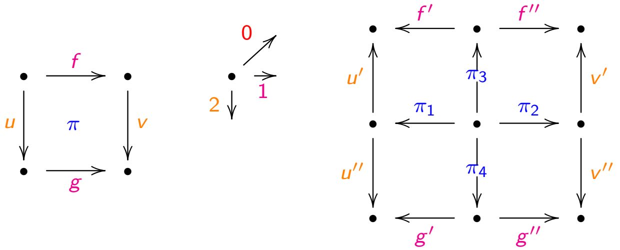
- for instance $\pi \colon \vee^2 \to \mathbf{Set}$ has four faces f, g, u, v (spans)



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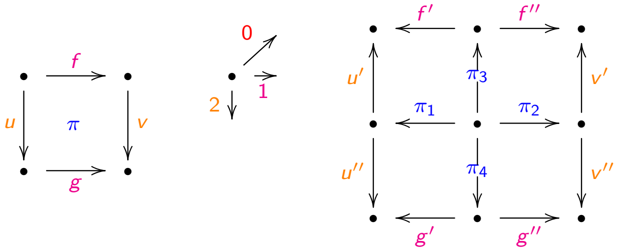

- transversal map $\varphi\colon x \to y\colon \vee^n \to \mathbf{Set}$ (a natural transformation)
- they compose strictly (in direction 0, the *transversal* direction)
- and give the comparisons for the i-composition of n-cubes.

11. Extending adjunctions to weak multiple categories

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(Span**Set** and Cosp**Set** have all multiple limits and colimits.)

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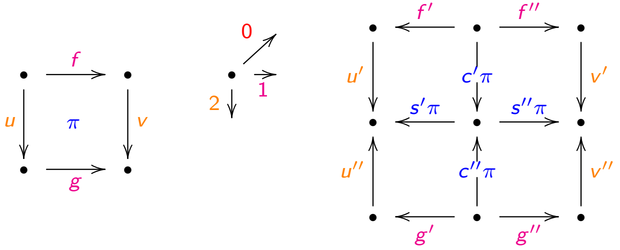

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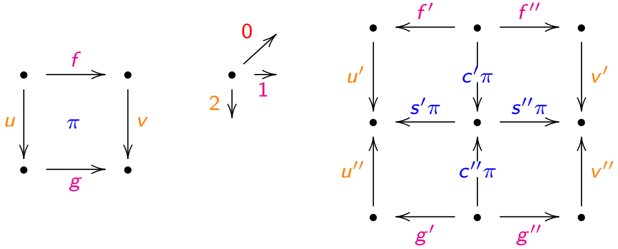

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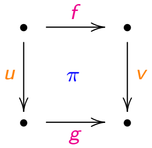

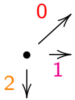

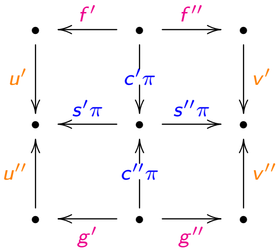

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Directed interchange for 1- and 2-directed compositions

$$\chi(\pi, \pi', \rho, \rho')\colon (\pi +_1 \pi') +_2 (\rho +_1 \rho') \to (\pi +_2 \rho) +_1 (\pi' +_2 \rho').$$

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in a double category of chiral categories, lax and colax functors.

14. Strict multiple categories (A. & C. Ehresmann)

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Multiple set A (in $\mathbf{Set}^{\underline{\mathbb{M}}^{\rm op}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}$
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Strict multiple category A:

a multiple set with (strict) compositions and interchange

- i-composition: $x +_i y$, for $i \in \mathbf{i}$, $x, y \in A_{\mathbf{i}}$, $\partial_i^+ x = \partial_i^- y$.

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in dimension 3, intercategories include:
duoidal categories, monoidal double categories, cubical bicategories, double bicategories, Gray categories

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The multiple site $\underline{\mathbb{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
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faces: $\partial_i^\alpha \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}$, $(\partial_i^\alpha t)(j) = t(j), \ \partial_i^\alpha(t)(i) = \alpha$

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(The cubical site has objects 2^n; its indices must be normalised.)

17. References for double and multiple categories

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